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Particle Behavior in Aesthetic Field Theory*

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Abstract

We review the principles of aesthetic field theory and the latest results obtained from computer studies of the equations.

1. *Review of Literature*

In the past, there have been some attempts at obtaining nonsingular particle-like solutions of nonlinear partial differential equations. Let me start off by giving a brief discussion of the kinds of results obtained by others, thus far.

(a) Rosen (1966) worked with the equation

$$\ddot{\theta} - \nabla^2\theta = 3g\theta^5 \tag{1.1}$$

He found a static solution

$$\theta = \frac{z}{(z^4 g + r^2)^{1/2}} \tag{1.2}$$

with z, g as parameters. The graph of this function has a maximum at the origin and goes to zero at infinity. The field equations are not drawn from physics. Nor is the solution particularly physical. The particle has little structure. Also, the solution describes only one particle.

(b) Anderson and Derrick (1970) worked with

$$\ddot{\theta} - \nabla^2\theta = g\theta^3 \tag{1.3}$$

They found a spherically symmetric solution with only a little more structure than Rosen.†

(c) Born and Infeld (1934) generalized electrodynamics in a nonlinear way.

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† See also Bisshopp, F. *International Journal of Theoretical Physics*, **11**, 5.

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Their equations are rather complicated so we will not repeat them here. They obtained a solution

$$E_r = \frac{q}{r_0^2} \frac{1}{\sqrt{1 + \left(\frac{r}{r_0}\right)^4}} \tag{1.4}$$

with q, r_0 as parameters. The shape of this particle is somewhat similar to Rosen, in that it has a maximum at the origin and goes to zero at infinity.

(d) Rosen (1972) worked with the classical Yang–Mills equation and found no particle-like solutions.

(e) Renada and Soler (1972) did find a particle-like solution, but they had to go to 5 dimensions.

(f) McLaughlin and Scott (1973) give reference to extensive work done in one spatial dimension equations. They refer to "pulse-like" solutions called solitons.

The list of references discussed above is representative (and not exhaustive) of what has been done by others. In summary, we may say that the results are rather limted, to say the least.

The first question in looking for particle-like behavior in a nonlinear theory is, which field equation should we use? We can't use Maxwell's equations or the Dirac equation or the Einstein gravitational equations, since here the particle is introduced by hand via the current density, the mass and charge, and the stress energy momentum tensor.

Some generalizations of existing equations that have been tried are:

(a) Born and Infeld's (1934) generalization of electromagnetism.

(b) Heisenberg's (1961) generalization of the Dirac equation.

(c) Einstein's (1955) and Schroedinger's (1950) generalizations of the gravitational equations.

The difficulties with such generalizations are:

(a) An infinite number of generalizations are possible. There is no experimental confirmation of any of the generalizations proposed. Thus, there is no reason to adopt any of these generalizations at this time.

(b) The equations are so complicated that it is difficult to say what the generalizations even imply.

(c) However, the difficulties go much deeper than this. For example, let us consider the Einstein–Schroedinger equations. They are obtained from an action principle, $\delta \int \mathscr{L} \sqrt{(-g)}\, d^4x = 0$. What they did is to arbitrarily exclude higher derivatives from the Lagrangian density. This was done so that the final equations are no higher than 2nd derivatives in g_{ij}.

But, there is nothing wrong with a higher derivative! We can say that an altogether too cavalier approach has been taken with respect to higher derivatives. Such a principle of exclusion of higher derivatives may have some local justification, but it is hard to consider it as anything but ad hoc. In summary the difficulty of the generalizations, thus far introduced, is that of mathematical inelegance.

The question is this: Is it possible to formulate a field theory based on mathematically aesthetic principles? What might these principles be? Let me list some principles that we have been working with. The list is not meant to be rigid.

2. *Aesthetic Principles*

(a) All derivatives are treated in a uniform way.

(b) All tensors are treated in a uniform way.

(c) The field is to be continuous, singularity free, possess a Taylor series expansion—in short be analytic.

(d) The field should go to zero at infinity. We refer to this condition as natural boundary conditions. Another possibility is that some fields be constant at infinity.

(e) The theory should be self consistent. By this we mean that the equations should be obtained using Aristotelian logic.

(f) No arbitrary functions should appear in the theory as this leads to too much arbitrariness. This excludes wave solutions, that have the function and the time derivative of the wave function arbitrary, on say a $t = 0$ hypersurface.

(g) There should exist certain coordinate transformations that do not lead to a dynamic effect. We have worked with $O(3)$, $O(4)$, $O'(3) \times T$, $L(4)$ (O refers to rotations, L to Lorentz transformations, T to time translations, and the prime means inhomogeneous transformations).

We feel these principles are of such an attractive nature that we would expect them to be incorporated into physics at a fundamental level.

We do not work in curved space for the following reasons:

(a) Flat space is simpler. One does not usually go to a more complicated theory unless the simpler theory has been shown to be inadequate. This has not been the case.

(b) Microscopic physics does not call for curved space, at least not yet.

(c) Gupta (1957) and others have shown that the results of gravitational theory can be obtained without curved space.

We shall work in a cartesian coordinate system. We refer to the 4th coordinate axis as the time axis.

3. *Derivation of Field Equations*

As most theories allow for a vector field, we shall start off by assuming the existence of a vector field A_i.

We write for the change of the vector between two neighboring points

$$dA_i = \Gamma^j_{ik} A_j \, dx^k \tag{3.1}$$

That is, the change of A_i should depend on the displacement between the two points dx^k. We drop terms of order $(dx)^2$ and higher. The change should also depend on A_j. The expression (3.1) can be made more general by allowing Γ^j_{ik} to be a function of A_i, among other things. We call the set of coefficients Γ^j_{ik} the change function as it determines the change dA_i.

For a 2nd vector B_i, we write

$$dB_i = \Gamma^j_{ik} B_j \, dx^k \tag{3.2}$$

We are, here, assuming that Γ^j_{ik} is a universal change function that determines the change of all vectors in a similar fashion. We are thus requiring that one set of numbers corresponding to a vector should not be fundamentally distinguished from any other vector set of numbers. After all, there is nothing painted on a vector that says it is more important than any other vector.

Going one step beyond this, we require that the change function determines the change of all tensor functions. An nth rank tensor acts like a product of n vectors in this regard.

For the product $A_i B_j$ we have, from (3.1) and (3.2),

$$d(A_i B_j) = (\Gamma^t_{ik} A_t B_j + \Gamma^t_{jk} A_i B_t) \, dx^k \tag{3.3}$$

A 2nd rank tensor acts like a product of two vectors. Thus, we get, by inspecting (3.3)

$$dg_{ij} = (\Gamma^t_{ik} g_{tj} + \Gamma^t_{jk} g_{it}) \, dx^k \tag{3.4}$$

Not only is the change function to determine the change of these quantities, it must also determine its own change as well. In cartesian coordinates the difference between two vectors, dA_i, is itself, a vector (unlike the case of curvilinear coordinates). Thus, Γ^i_{jk} is a 3rd rank tensor.

A^i is introduced via $A_i = g_{ij} A^j$. From (3.1) and (3.4), and assuming, for now, that g_{ij} has an inverse at all points, we get

$$dA^i = -\Gamma^i_{jk} A^j \, dx^k \tag{3.5}$$

Since Γ^i_{jk} acts like $A^i B_j C_k$, we get from (3.1), (3.2) and (3.3)

$$d\Gamma^i_{jk} = (\Gamma^i_{mk} \Gamma^m_{jl} + \Gamma^i_{jm} \Gamma^m_{kl} - \Gamma^m_{jk} \, \Gamma^i_{ml}) \, dx^l \tag{3.6}$$

From continuity we have

$$d\Gamma^i_{jk} = \frac{\partial \Gamma^i_{jk}}{\partial x^l} \, dx^l \tag{3.7}$$

Thus, we get for our field equations

$$\frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^m_{jk} \Gamma^i_{ml} - \Gamma^i_{mk} \Gamma^m_{jl} - \Gamma^i_{jm} \Gamma^m_{kl} = 0 \tag{3.8}$$

These are 256 nonlinear, partial differential equations for the change function alone.

We note that the derivatives of Γ^i_{jk} are given by products of gammas. If we take a derivative of this equation, we see that the second derivative of Γ^i_{jk} is also expressed entirely in terms of products of gammas. In a similar fashion we can show that all derivatives of Γ^i_{jk} are given by products of gammas. We shall make use of this result later on.

Let us summarize what we have, thus far. We have assumed that there exists a change function that determines the change of all functions (see footnote one). But the change function is itself a function. Therefore, by Aristotalian logic the change function must determine its own change as well. The functions we are dealing with are tensor functions. This is the self consistent requirement listed under Aesthetic Principles (principle e).

For g_{ij}, we have

$$\frac{\partial g_{ij}}{\partial x^k} - \Gamma^t_{ik} g_{tj} - \Gamma^t_{jk} g_{it} = 0 \qquad (3.9)$$

From (3.1) and (3.5), we see that $A_i A^i$ is constant. In addition we find that all scalars, that can be constructed from products and contractions of the fields, are constant.

For a set of basis vectors $e^\alpha{}_i$, we get

$$de^\alpha{}_i = \Gamma^j_{ik} e^\alpha{}_j \, dx^k \qquad (3.10)$$

Here, α denotes which basis vector we are dealing with. From continuity, we have

$$\frac{\partial e^\alpha{}_i}{\partial x^k} = \Gamma^j_{ik} e^\alpha{}_j \qquad (3.11)$$

Introducing a set of dual vectors $e_\alpha{}^i$ satisfying

$$e_\alpha{}^i e^\alpha{}_j = \delta_j{}^i \qquad (3.12)$$

we get, from (3.11)

$$\Gamma^i_{jk} = e^\alpha{}_j \frac{\partial e^\alpha{}_j}{\partial x^k} \qquad (3.13)$$

Thus, here Γ^i_{jk} is a function of basis vectors as previously suggested (see discussion under equation (3.1)). From (3.16) and (3.12) we get

$$de_\alpha{}^i = -\Gamma^i_{jk} e^\beta{}_i \, dx^k \qquad (3.14)$$

This has the same structure as dA^j in (3.5). The result (3.14) may be used in the case where the determinant of g is zero and so the inverse field g^{ij} does not exist. The utility of (3.14) would then depend on $e_i{}^\alpha$ having an inverse at all points.

A variant theory is given by

$$de^\alpha{}_i = (\Gamma^j_{ik} e^\alpha{}_j - \bar{\Gamma}^\alpha_{\beta k} e^\beta{}_i) \, dx^k \qquad (3.15)$$

Here we have a change function for ij indices and a change function for $\alpha\beta$ indices. Since Γ^i_{jk} itself has only ij indices, we get the same final equations as previously [equation (3.8)].

[1] The manner in which the change function determines the change of various functions is such that all tensors and derivatives of tensors are treated in a uniform way.

Thus, we have formulated a theory in which all tensors are treated in a uniform manner. The question that arises is whether this leads to a never ending series of restrictions on the theory.

For example, let us consider $d(\Gamma^i_{jk}g_{tl})$. This acts like $A^iB_jC_kD_tE_l$. Thus, from (3.1), (3.2) and (3.3) we get

$$d\Gamma^i_{jk}g_{tl} = (-\Gamma^m_{jk}\Gamma^i_{ms}g_{tl} + \Gamma^i_{mk}\Gamma^m_{js}g_{tl}$$

$$+ \Gamma^i_{jm}\Gamma^m_{ks}g_{tl} + \Gamma^i_{jk}\Gamma^m_{ts}g_{ml} + \Gamma^i_{jk}\Gamma^m_{ls}g_{tm})\,dx^s \tag{3.16}$$

But, on the other hand, we also have

$$d(\Gamma^i_{jk}g_{tl}) = (d\Gamma^i_{jk})g_{tl} + \Gamma^i_{jk}(dg_{tl}) \tag{3.17}$$

Inserting (3.4) and (3.6) into (3.17), we get the same answer as (3.16). We can do this for a general product of fields (including contractions). The result is that we get no new restrictions on the theory by treating all tensors in a uniform way.

What about the case of derivatives? For example, we may consider $d(\partial_m\Gamma^i_{jk})$. But by the field equation, what is inside the parenthesis, can be expressed in terms of products of the fields. As we pointed out previously, this is the case for all derivatives of the fields (this is true whether they be of A_i, $e^\alpha{}_j$, g_{ij} or Γ^i_{jk}). Thus, by the results of the previous paragraph, we conclude that no new conditions appear as a result of requiring that all derivatives are treated in a uniform way.

Note also that a derivative acts like a vector in a cartesian space. Thus, the condition that all derivatives are treated in a uniform way can be included in the hypothesis that requires all tensors be treated uniformly so far as their changes are concerned.

4. *Local Existence of Solutions*

Thus, we have constructed a field theory based on mathematically aesthetic principles. This may be all well and good, but suppose there are no nontrivial solutions to the field equations. So the next problem is to demonstrate local existence of solutions. We will be unable to prove global existence analytically. However, our computer work suggests that global existence may also be present.

Let us digress for the moment and say something about the computer program as it will give us some intuitive feeling for what is going on.

We have from (3.6)

$$\Gamma^i_{jk}(Q) = \Gamma^i_{jk}(P) + \{\Gamma^i_{mk}\Gamma^m_{jl} + \Gamma^i_{jm}\Gamma^m_{kl} - \Gamma^m_{jk}\Gamma^i_{ml}\}\,dx^l \tag{4.1}$$

where Q and P are two neighboring points. The bracketed quantity to the lowest order is evaluated at P. In the computer we have a finite grid Δx^l instead of the infinitesimal dx^l. From (3.17) we may calculate the field at Q given the field at P. Once we have the field at Q, we can calculate the field at the next neighboring point Q in the same way, since the field equations for Γ^i_{jk} hold at all points. In such a manner, we can map out the field at all points of space given the field at one particular point. This mapping process can be carried out in principle no matter how small the grid is. The only problem is whether we get the same answer at a point if we arrive at that point along different paths. In order for the answer to be unique it is necessary for a set of integrability equations to be satisfied. These integrability equations arise from the requirement that all mixed derivatives of fields be symmetric. Then, the results will be independent of path.

We can look at the problem in a somewhat different way. Given a finite field Γ^i_{jk} at P, we can calculate a finite number of derivatives of Γ^i_{jk} at P, since derivatives of Γ^i_{jk} are given by products of Γ^i_{jk}. Also since a second derivative is given by the difference of first derivatives at neighboring points, we can solve for $\partial\Gamma^i_{jk}/\partial x^l$ at Q in terms of the fields at P. In this way, a finite number of derivatives of Γ^i_{jk} can be seen to exist in the region around P (see footnote 2). Thus, the conditions for the existence of a Taylor's series are met (see T. Apostal, Mathematical Analysis, p. 96). Thus, we have

$$\Gamma^i_{jk}(R) = \Gamma^i_{jk}(P) + \left.\frac{\partial\Gamma^i_{jk}}{\partial x^l}\right)_P \Delta x^l + \frac{1}{2}\left.\frac{\partial^2\Gamma^i_{jk}}{\partial x^m \partial x^n}\right)_P \Delta x^m \Delta x^n + \ldots \tag{4.2}$$

where Δx^l is the displacement between R and P. From (4.2) and (3.8) we see that we can calculate $\Gamma^i_{jk}(R)$ in a unique manner given $\Gamma^i_{jk}(P)$, provided that the mixed derivatives in (4.2) are symmetric at P. This agrees with what we have said previously. From

$$\frac{\partial^2 g_{ij}}{\partial x^k \partial x^m} = \frac{\partial^2 g_{ij}}{\partial x^m \partial x^k} \tag{4.3}$$

we get

$$g_{th} R^t_{imk} + g_{it} R^t_{hmk} = 0 \tag{4.4}$$

where

$$R^t_{imk} = \frac{\partial\Gamma^t_{ik}}{\partial x^m} - \frac{\partial\Gamma^t_{im}}{\partial x^k} - \Gamma^j_{im}\Gamma^t_{jk} + \Gamma^j_{ik}\Gamma^t_{jm} \tag{4.5}$$

From

$$\frac{\partial^2\Gamma^i_{jk}}{\partial x^p \partial x^l} = \frac{\partial^2\Gamma^i_{jk}}{\partial x^l \partial x^p} \tag{4.6}$$

we get

$$\Gamma^i_{mk} R^m_{jpl} + \Gamma^i_{jm} R^m_{kpl} - \Gamma^m_{jk} R^i_{mpl} = 0 \tag{4.7}$$

[2] An infinite derivative would involve an infinite number of gamma terms, and it is not clear whether this would be finite. Thus, we have not been able to prove global existence analytically.

From (using (3.4))

$$\frac{\partial^2 e^{\alpha}{}_i}{\partial x^j \partial x^k} = \frac{\partial^2 e^{\alpha}{}_i}{\partial x^k \partial x^j} \tag{4.8}$$

we get

$$R^t_{imk} = 0 \tag{4.9}$$

Thus, all the conditions above are satisfied provided $R^t_{imk} = 0$.

Furthermore, if we take

$$\frac{\partial^2 (\Gamma^s_{tp} \Gamma^l_{jf})}{\partial x^j \partial x^k} = \frac{\partial^2 (\Gamma^s_{tp} \Gamma^l_{jf})}{\partial x^k \partial x^j} \tag{4.10}$$

we see that this is identically satisfied provided (4.6) is satisfied (which is the case if $R^t_{imk} = 0$). Similarly, we see that for all products of the field (including contractions) the mixed derivatives are symmetric once $R^i_{jkl} = 0$.

We thus get one set of integrability equations for all tensor fields.

Let us summarize where we stand. We shall abbreviate equation (3.8) as $\Gamma^i_{jk;l} = 0$, equation (3.9) as $g_{ij;k} = 0$, equation (3.10) as $e^{\alpha}_{i;k} = 0$. The semicolon here is an abbreviation for terms that have a formal similarity with terms in a covariant derivative. We have to emphasize that this similarity is only formal, since we are dealing with a heirarchy of tensors in a cartesian space and we are not talking about curvilinear coordinates.

The basic field equations of a aesthetic field theory are then

$$T^{ij\ldots}_{kmpl\ldots;s} = 0 \tag{4.11}$$

where $T^{ij\ldots}_{kmpl\ldots}$ is any well defined combination of $\Gamma^i_{jk}, g_{ij}, e^{\alpha}{}_i, \partial_m$. When we are considering Γ^i_{jk} itself, then we end up with the basic field equations (3.8). In conjunction with this equation we need

$$R^i_{jkl} = 0 \tag{4.12}$$

which from (3.8) we see is a restriction on the initial data at an arbitrary origin point. Equation (4.11) represents an infinite number of equations. However, we already argued that this did not lead to an increased set of restrictions on the initial data.

The only other way we can think of treating all infinite derivatives of the field in a uniform way would be to have an equation involving an infinite number of derivatives. It seems to us that there are advantages in the approach we have taken.

Should we consider the variant theory based on (3.15), then we would have to replace (4.12) by (4.7) and (4.4).

We prove, here, that if R^i_{jkl} is zero at one point, then it must be zero at all points. R^i_{jkl} acts like $A^i B_j C_k D_l$. Thus, we have

$$\frac{\partial R^i_{jkl}}{\partial x^p} = -\Gamma^i_{mp} R^m_{jkl} + \Gamma^m_{jp} R^i_{mkl} + \Gamma^m_{kp} R^i_{jml} + \Gamma^m_{lp} R^i_{jkm} \tag{4.13}$$

If $R^i_{jkl} = 0$ at the origin, then $\partial R^i_{jkl}/\partial x^p$ is zero from (4.13). If we then take a derivative of (4.13), $\partial^2 R^i_{jkl}/\partial x^p \partial x^t$ must be zero as well. Similarly we can show that all derivatives of R^i_{jkl} are zero at the origin. Thus, R^i_{jkl} is zero at all points if it is zero at one point. This result may appear quite reasonable. However, we have been able to construct theories where this is not the case (Muraskin, 1973, 1975) by what amounts to judicious use of scalar functions.

Using (3.8) and (4.9), we get

$$\Gamma^t_{ij}\Gamma^i_{km} - \Gamma^t_{ij}\Gamma^j_{mk} + \Gamma^i_{im}\Gamma^t_{jk} - \Gamma^i_{ik}\Gamma^t_{jm} = 0 \tag{4.14}$$

This is then the form that the integrability equations take. They clearly act as restrictions on the initial data. These equations are 96 in number (there is antisymmetry between k and m) which is more than the number of Γ^i_{jk} at the origin, which is 64. Thus, it is not clear at this point that nontrivial solutions to the field theory exist.

However, by now we have found literally hundreds of solutions to (4.14) (or 4.7). We shall write down a particular solution later on. The conclusions is that solutions to the field theory exists locally. Thus, the theory does say something. The problem then is to find just what sort of information is contained in the theory.

By writing

$$\Gamma^i_{jk} = a_\alpha{}^i a^\beta{}_j a^\gamma{}_k \Gamma^\alpha_{\beta\gamma} \tag{4.15}$$

we find (4.14)

$$\Gamma^\alpha_{\beta\gamma}\Gamma^\lambda_{\gamma\rho} - \Gamma^\alpha_{\beta\lambda}\Gamma^\lambda_{\rho\gamma} + \Gamma^\lambda_{\beta\rho}\Gamma^\alpha_{\lambda\gamma} - \Gamma^\lambda_{\beta\gamma}\Gamma^\alpha_{\lambda\rho} = 0 \tag{4.16}$$

It follows that if we find a simple solution to (4.16) for $\Gamma^\alpha_{\beta\gamma}$, we can get a more complicated Γ^i_{jk} by making use of an arbitrary $a^\alpha{}_i$. The resulting Γ^i_{jk} will automatically satisfy (4.14). Thus, from a simple solution $\Gamma^\alpha_{\beta\gamma}$, we can generate more complicated solutions of the integrability equations.

5. *Results*

(a) We have found bounded particle solutions. A particle is taken to be a maximum or minimum in a field component.

(b) We have found solutions for which no sign of singularities appear anywhere.

(c) We have found solutions consistent with the natural boundary conditions.

(d) We have observed a two particle collision. The location of the maximum (minimum) center as a function of time is not along a straight line.

(e) Sinusoidal behavior along a coordinate axis has been observed (although integrability was not satisfied here).

One of our particle systems is shown in Figure 1. Another in Figure 2. Actually, Figure 1 is from the closely related equation $\Lambda_{ijk;l} = 0$. However, similar type results have also been obtained for $\Gamma^i_{jk;l} = 0$. Notice there is no sign of singularities building up in these maps. Also the field tends towards zero outside the particle system.

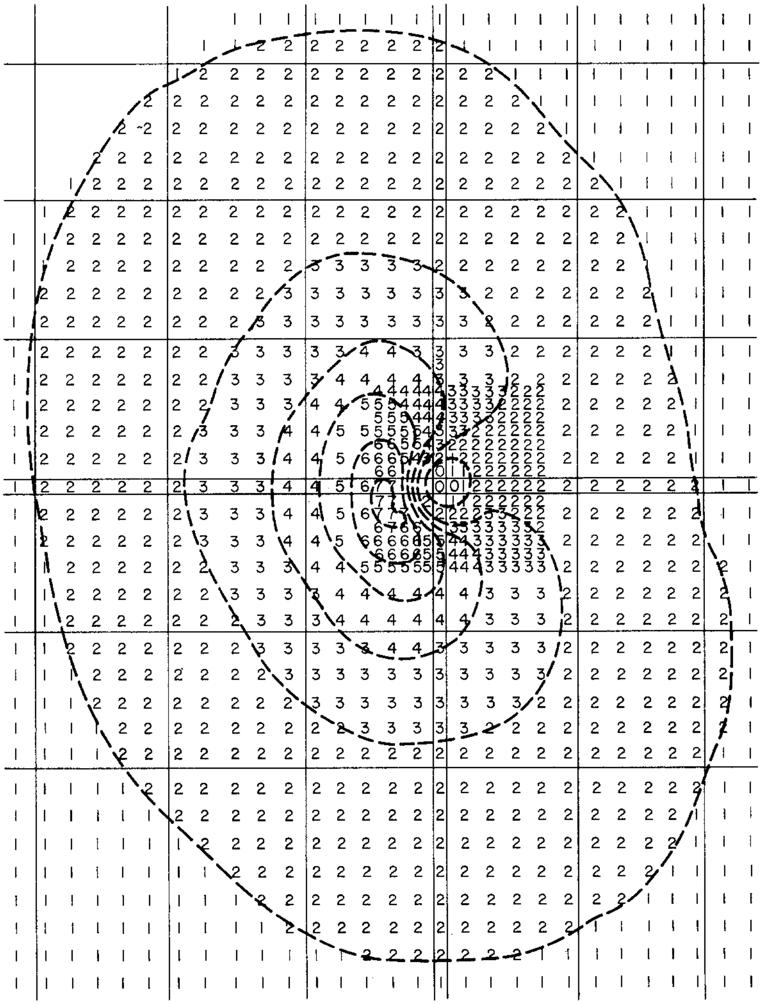

Figure 1—Map of g_{00} for first set of data obtained by us which leads to a bounded particle system.

6. *Computer Studies*

We get a tremendous amount of numbers coming off the computer. How do we know that nonsence is not being produced? What we do is make use of one of our previous results. We recall that the integrability equations hold at all points, if they hold at one point. Thus, we may periodically check whether the 96 integrability equations (4.14) (or the 384 equations (4.7)) are satisfied. It is too much to expect that these equations would be zero to 9, 10, 11, or 12 places by accident.

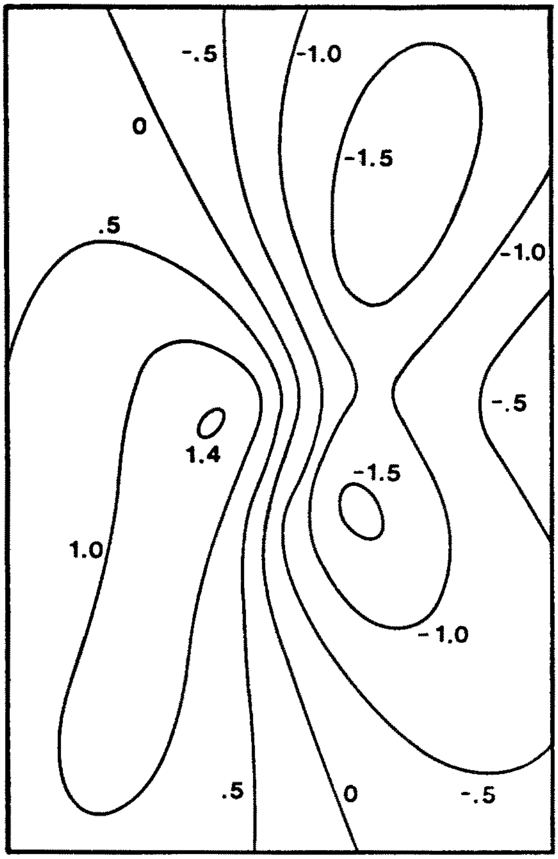

Figure 2—Map of Γ^1_{11} for data leading to maximum (minimum) center not lying on a straight line as a function of time.

If we make the grid small enough, we can get quite accurate results. We have, in addition, employed a scheme where we subtract off corrections that arise from repeatedly halving the grid. We can get agreement for $(\Gamma^i_{jk})_{\text{corrected}} - (\Gamma^i_{jk})^{1/2\text{grid}}_{\text{corrected}}$ to say 11 or 12 places for the case of the data appearing in Figure 1 and Figure 2. We could get even better agreement if one wanted to increase the running time.

The type of results we get are strongly dependent on the form of the initial data at the origin. For a long time, we were getting fields growing bigger and bigger as we moved away from the origin. In one instance we ran 48 straight hours on the computer with no sign of let up. This suggested to us that a singularity was developing. In fact, we can show analytically that for certain data choices singularities do occur—for example, if all Γ^i_{jk} are equal.

A set of data giving rise to a bounded particle system is as follows. We write

$$\Gamma^i_{jk} = a_\alpha{}^i a^\beta{}_j a^\gamma{}_k \Gamma^\alpha_{\beta\gamma} \tag{6.1}$$

with $\Gamma^\alpha_{\beta\gamma}$ having the following nonzero components

$$\begin{aligned}
&\Gamma^1_{10} = \Gamma^2_{20} = \Gamma^3_{30} = \Gamma^0_{00} = \Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} = A\\
&\Gamma^0_{11} = \Gamma^0_{22} = \Gamma^0_{33} = -B\\
&\Gamma^2_{13} = \Gamma^3_{21} = \Gamma^1_{32} = -\Gamma^1_{23} = -\Gamma^3_{12} = -\Gamma^2_{31} = C
\end{aligned} \tag{6.2}$$

For any choice of A, B, C we get the integrability equations (4.7) are satisfied. If $A = B = C$, then $R^i_{jkl} = 0$ is satisfied. $\Gamma^\alpha_{\beta\gamma}$ has the property of being unchanged by a 3-dimensional rotation. This can be seen from the fact that $\Gamma^\alpha_{\beta\gamma}$ has the structure ($\epsilon_{\rho\sigma\beta\gamma}$ is the antisymmetric symbol)

$$\Gamma^\alpha_{\beta\gamma} = \delta_\beta{}^\alpha \phi_\gamma - g_{\beta\gamma}\psi^\alpha + \delta_\gamma{}^\alpha \theta_\beta + g^{\alpha\sigma} B^\rho \epsilon_{\rho\sigma\beta\gamma} + \mu^\alpha \lambda_\beta \chi_\gamma \tag{6.3}$$

with

$$\boldsymbol{\varphi} = \boldsymbol{\psi} = \boldsymbol{\theta} = \mathbf{B} = \boldsymbol{\mu} = \boldsymbol{\lambda} = \boldsymbol{\chi} = 0$$

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad g^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.4}$$

since $g_{\alpha\beta}, g^{\alpha\beta}; \phi_0, \theta_0 \ldots$ are unchanged by a 3-dimensional rotation, it follows that $\Gamma^\alpha_{\beta\gamma}$ is invariant under such a transformation.

In (6.1) $a^\alpha{}_i$ can be considered a dynamical field. $a^\alpha{}_i$ also has the same formal effect as a coordinate transformation. We may argue that there should exist a set of coordinate transformations that do not have the effect of introducing any dynamics. With the present data rotations of the coordinates have this property.

In this way, we build into the theory the seventh hypothesis (g), listed previously under Aesthetic Principles.

The set of data described above was our first example of a bounded particle solution to the equations $\Gamma^i_{jk;l} = 0$. Since then, we have found a solution with greater complexity. A planar map of this new solution is given in Figure 2. The data here does not appear (at least on the surface) to be related to a rotationally invariant $\Gamma^\alpha_{\beta\gamma}$. Still a rotation of coordinates does not appear to alter the overall characteristics of the particle system. Thus, strict invariance of $\Gamma^\alpha_{\beta\gamma}$ may not be necessary in order to satisfy the seventh aesthetic principle (g).

7. *Additional Points*

There are some points that we feel should perhaps be emphasized. In standard tensor analysis, there is no difference between covariant and contravariant indices in an orthogonal coordinate system. However, our introduction of upper and lower indices has a different purpose. In our approach, the coordinate system is just an arena for the dynamical fields. If we consider a field g_{ij}, we can introduce a dual or inverse field g^{jk} such that $g_{ij}g^{jk} = \delta_i^{\ k}$. Thus g^{ij} is just the cofactor of g_{ij} divided by the determinant of g_{ij}. In other words, g^{jk} is just an abbreviation for certain combinations of the field components g_{ij}. Thus, the upper indices carry dynamical information and does not represent parallel projection of a vector on the coordinate axes as distinguished from perpendicular projection.

Thus, we may introduce upper indices in a cartesian coordinate system. Fields with upper indices carry dynamical information. The problem to be concerned with is that g^{ij} may not exist at all points. In our initial work on aesthetic field theory, we assumed that $g_{ij} \to (-1, -1, -1, +1)$ at infinity and we supposed that g^{ij} could be defined at all points. However, in such a theory it was necessary for an infinite number of restrictions to be imposed on Γ^i_{jk} at the origin point in order for $\Gamma^i_{jk} \to 0$ at infinity. All invariants formed from Γ^i_{jk} and g_{ij}, g^{tm} had to be zero at the origin. Such a set of initial data is not easy to come by. Nevertheless, we did find some examples of such data. However, in none of these examples did we find a bounded solution.

Our best computer results occur with data that appear to satisfy $g_{ij} \to 0$ at infinity. Since $g \to 0$ at infinity, we conclude that g^{ij} is not defined at all points. Thus, we have to be careful and not introduce inverse fields when they are not defined.

We can still introduce upper indices using the dual field $e_\alpha{}^i$. The dual field can be introduced if it can be demonstrated that $e^\alpha{}_i \to \delta^\alpha{}_i$ at infinity.

On the other hand, once we have established the point that upper indices have dynamical character, we can argue that the introduction of upper indices, in terms of inverse fields for g_{ij} and $e^\alpha{}_i$ is not a necessity.

The difference between two vectors in a cartesian space is a vector. The role of the upper indices in (3.1) is to denote scalar products. This is a dynamical way for introducing a scalar product. We make the requirement that Γ^t_{tk} (as well as Γ^t_{kt}) act like a vector. This fixes the change of upper index fields to have the same structure as (3.5) (see footnote 3).

Up to now, we have assigned Γ^i_{jk} as well as g_{ij} in an arbitrary fashion at the origin point. But so far as the field equations are concerned, g_{ij} is a secondary type field. That is, Γ^i_{jk} determines the change of g_{ij}, but g_{ij} does not affect the change of Γ^i_{jk}. Thus, a simplifying hypothesis would be to consider only Γ^i_{jk} as arbitrary at the origin point. All tensors would be products or contractions of the basic field Γ^i_{jk}.

From Γ^i_{jk} we can form objects like $\bar{g}_{ij} \equiv \Gamma^t_{ti}\Gamma^s_{sj}$. However, since Γ^i_{jk} should go to zero at infinity, $\bar{g}_{ij}$ would also go to zero at infinity. Thus, $\bar{g}^{ij}$ would not be defined at all points. This tells us that we cannot indiscriminately introduce inverse fields once we impose the conditions $\Gamma^i_{jk} \to 0$ at infinity. Indeed, there is no g^{ij} constructed from Γ^i_{jk} that is defined at all points.

[3] Γ^t_{tk} acts like $A^tB_tC_k$ and like D_k.

With such a viewpoint, it is not necessary for Γ^i_{jk} to obey an infinite number of restrictions at the origin in order for $\Gamma^i_{jk} \to 0$ to go to zero at infinity. In fact, the data associated with Figure 2 is an example of a situation where Γ^i_{jk} appears to go to zero at infinity without any restrictions imposed at the origin.

8. *Outlook*

The aesthetic principles discussed here (see also Muraskin, 1970; 1971a, b; 1972a, b, c; 1973a, b; 1974; 1975; Muraskin and Clark, 1970; Muraskin and Ring, 1972; 1973; 1974a, b;) are meant to be of such a basic and attractive character that they would be desirable in any fundamental physical theory. Conversely, a theory that does not, for example, treat all derivatives in a uniform way, would appear to us to be incorrect. We have seen that a theory based on aesthetic principles does have considerable content. It is also hard to believe that solutions to the aesthetic equations with even greater complexity do not exist.

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